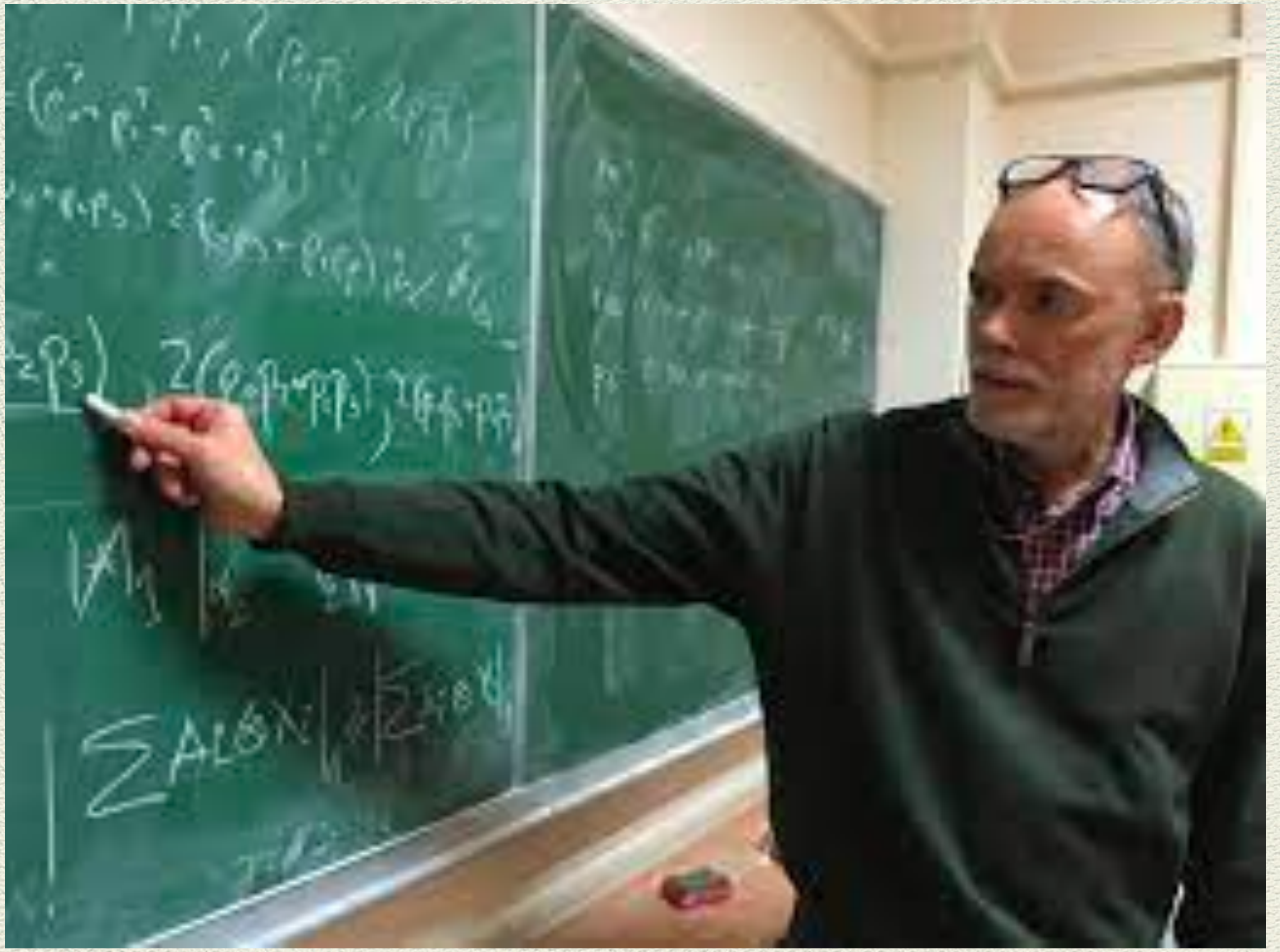
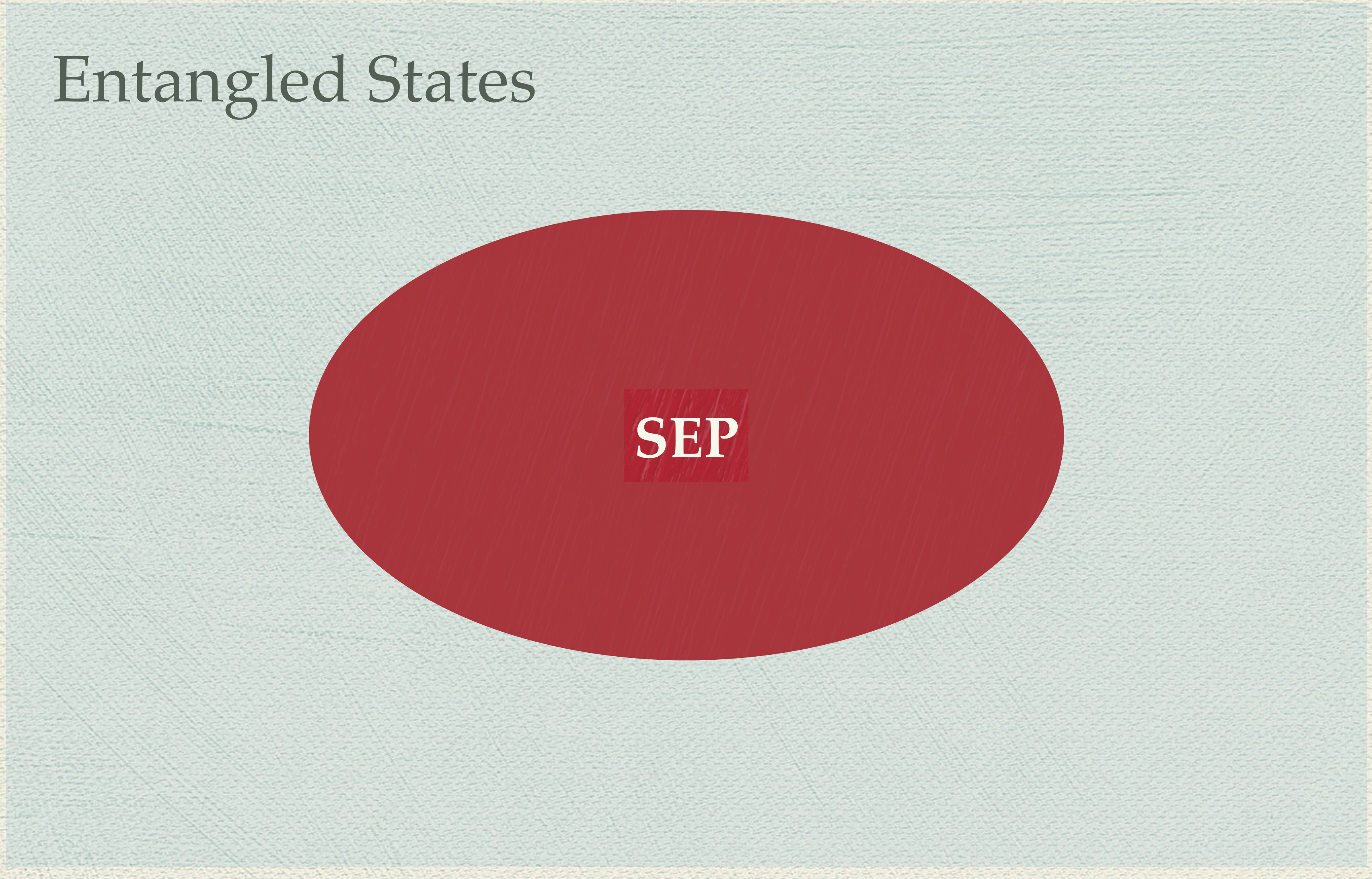
Entanglement Witnesses

Barbara Terhal



Dariusz Chruscinski

Separable States



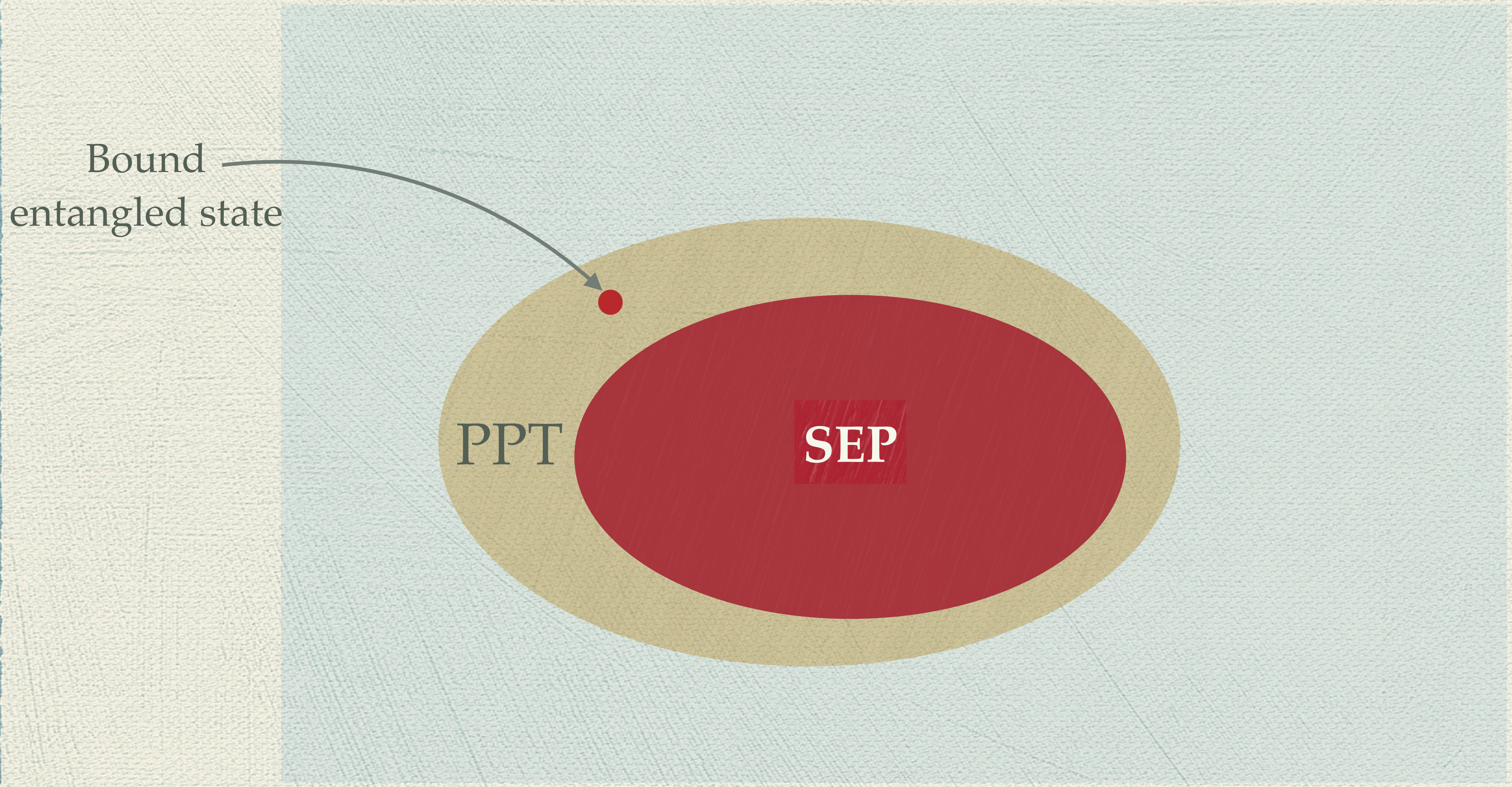
$$\rho^{sep} = \sum_i P_i \, \rho_i \otimes \sigma_i$$

Is this state separable?

$$\rho = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & p+1 & -2p & 0 \\ 0 & -2p & p+1 & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}$$

Positive Partial Transpose (PPT)

$$\rho = \sum_i P_i \, \rho_i \otimes \sigma_i \longrightarrow \rho^\Gamma \equiv (I \otimes T)\rho = \sum_i P_i \, \rho_i \otimes \sigma_i^T$$

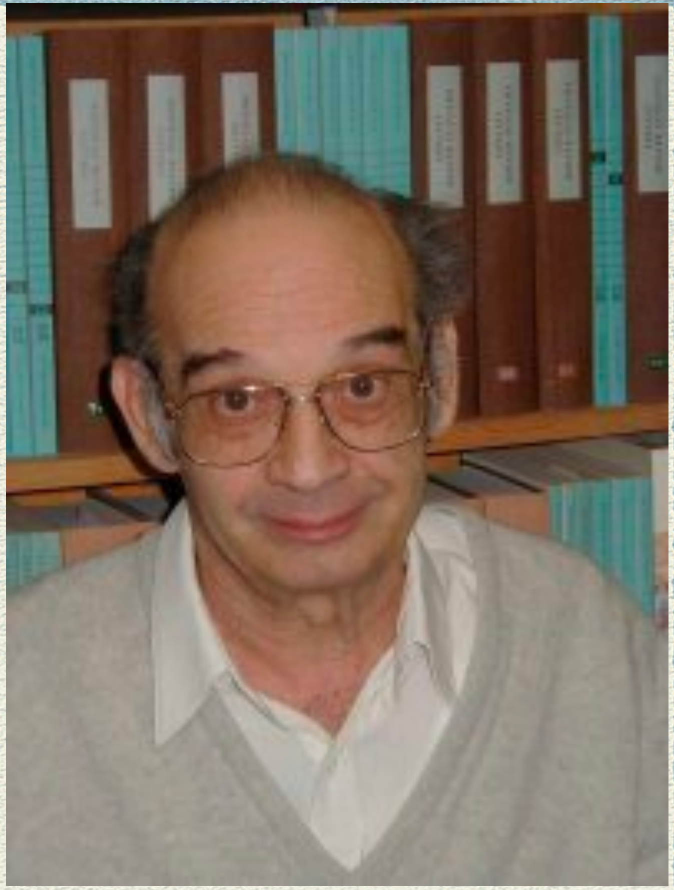


Peres Criteria

$$\rho = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & p+1 & -2p & 0 \\ 0 & -2p & p+1 & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}$$

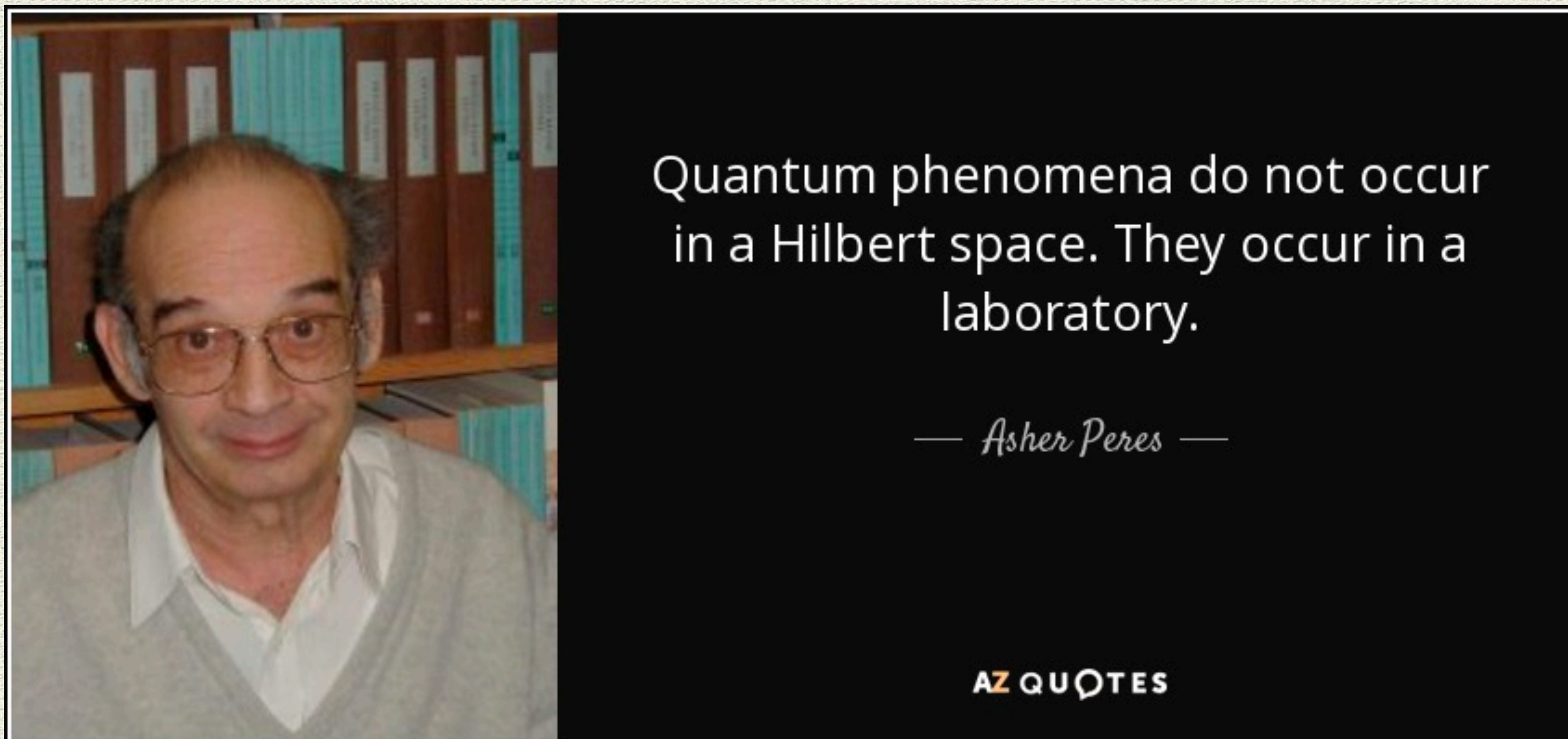
$$\rho^{\Gamma} = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & -2p \\ 0 & p+1 & 0 & 0 \\ 0 & 0 & p+1 & 0 \\ -2p & 0 & 0 & 1-p \end{pmatrix}$$

$$\lambda_{-} = \frac{1}{4}(1-3p)$$



Asher Peres

1934-2005


Quantum phenomena do not occur in a Hilbert space. They occur in a laboratory.
— Asher Peres —
AZ QUOTES

"Never underestimate the ingenuity of experimental physicists."

- Asher Peres, Found. Phys. 14, 1131 (1984)

Physicist and quantum information theory pioneer, Asher Peres (1934-2005) explored connections between quantum mechanics and the theory of relativity.

#quantumchangers

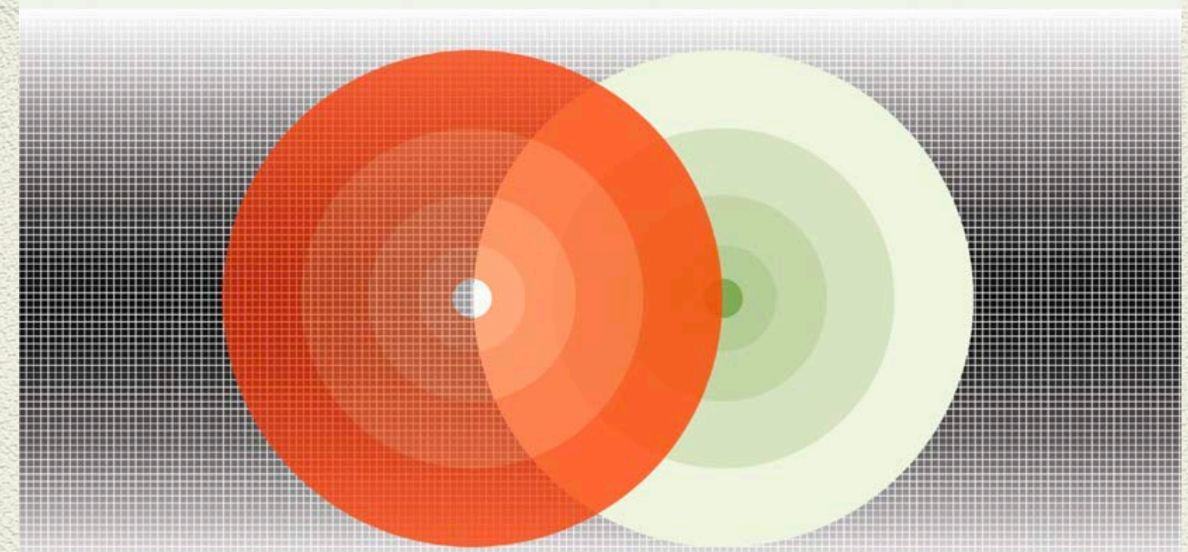
UNIVERSITY OF WATERLOO | IQC Institute for Quantum Computing

Quantum Theory: Concepts and Methods

by

Asher Peres

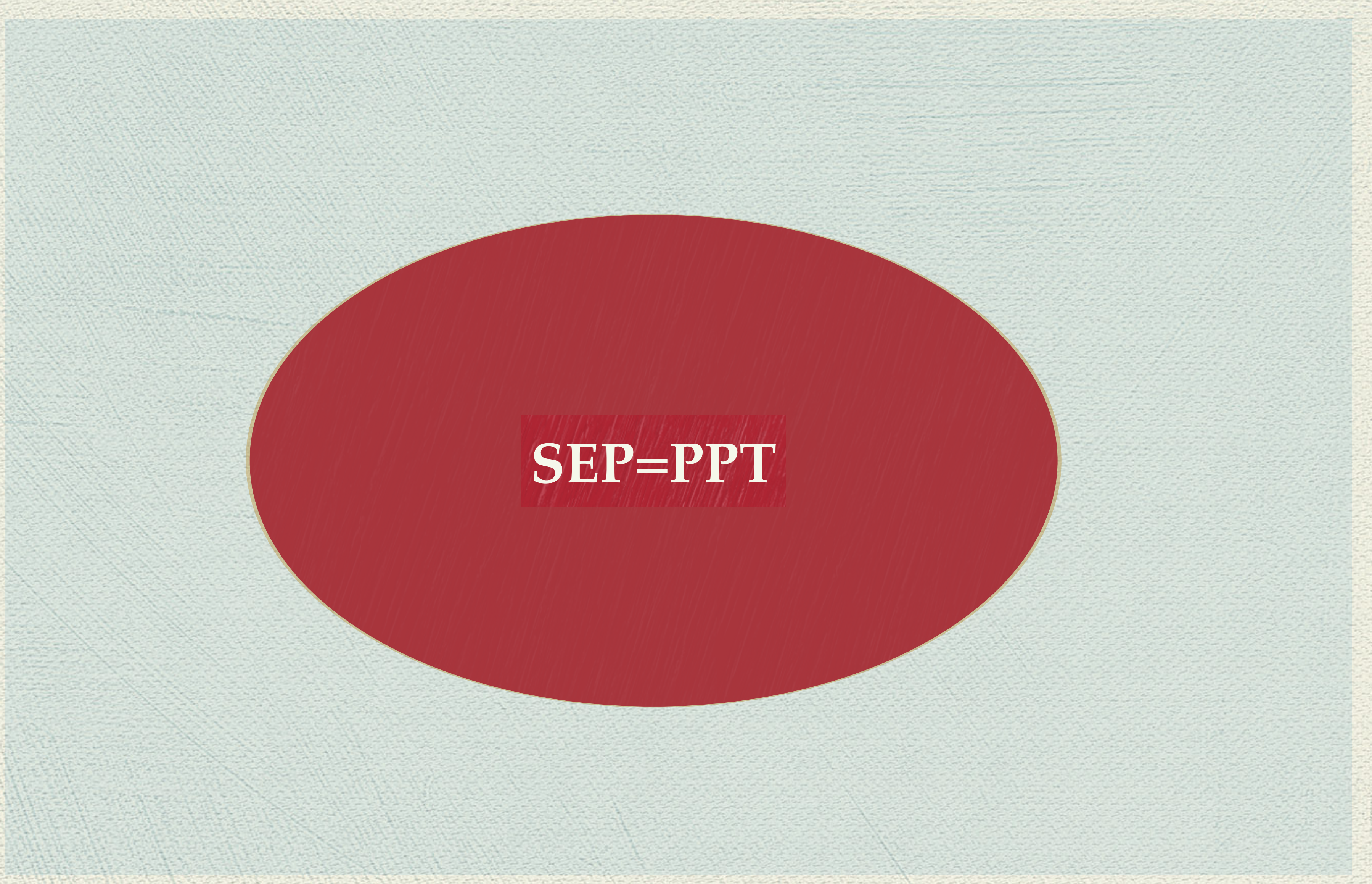
Springer Science+Business Media, LLC



Fundamental Theories of Physics

In 2×2 *and* 2×3 *dimensions*

PPT=SEP

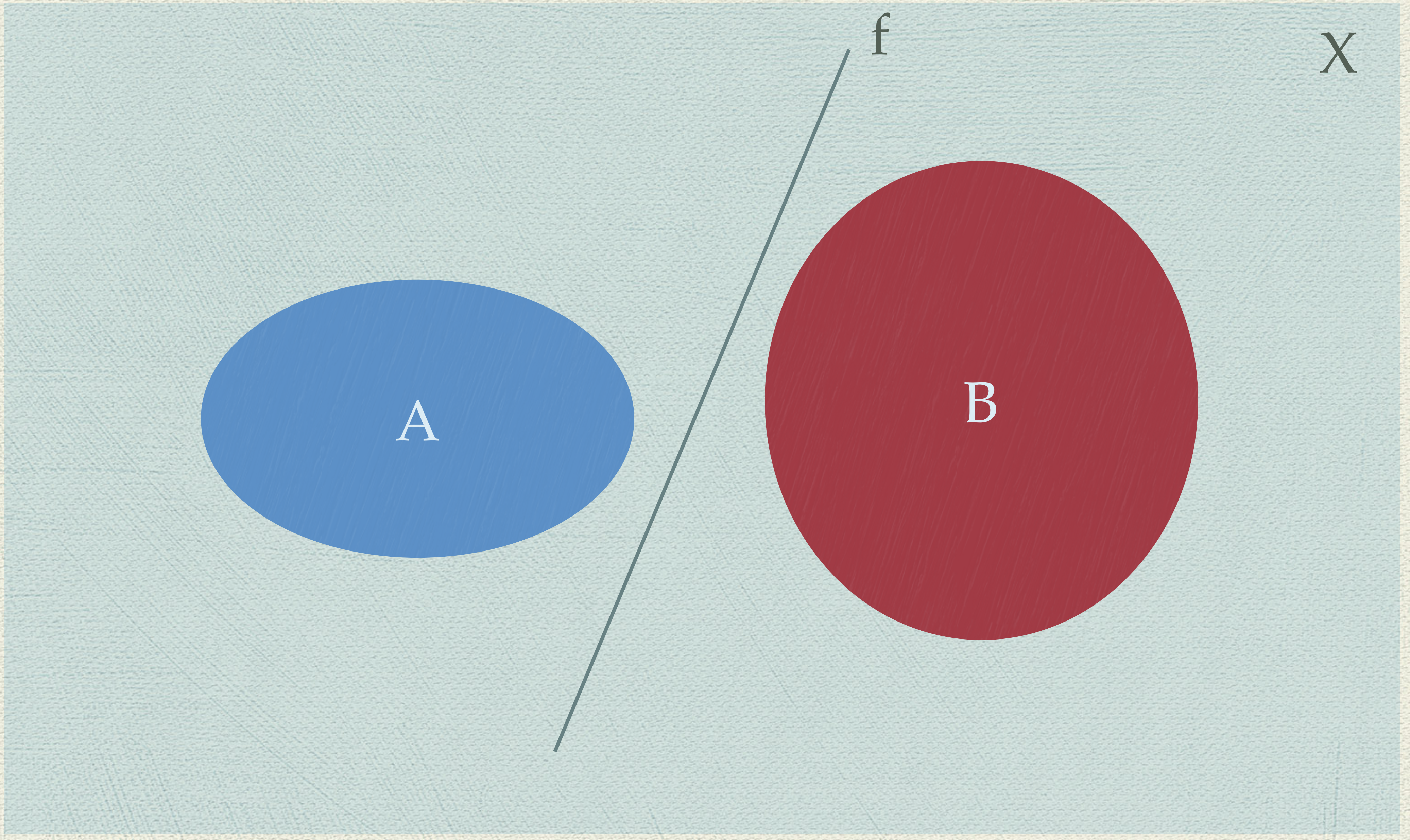


Entanglement Witnesses

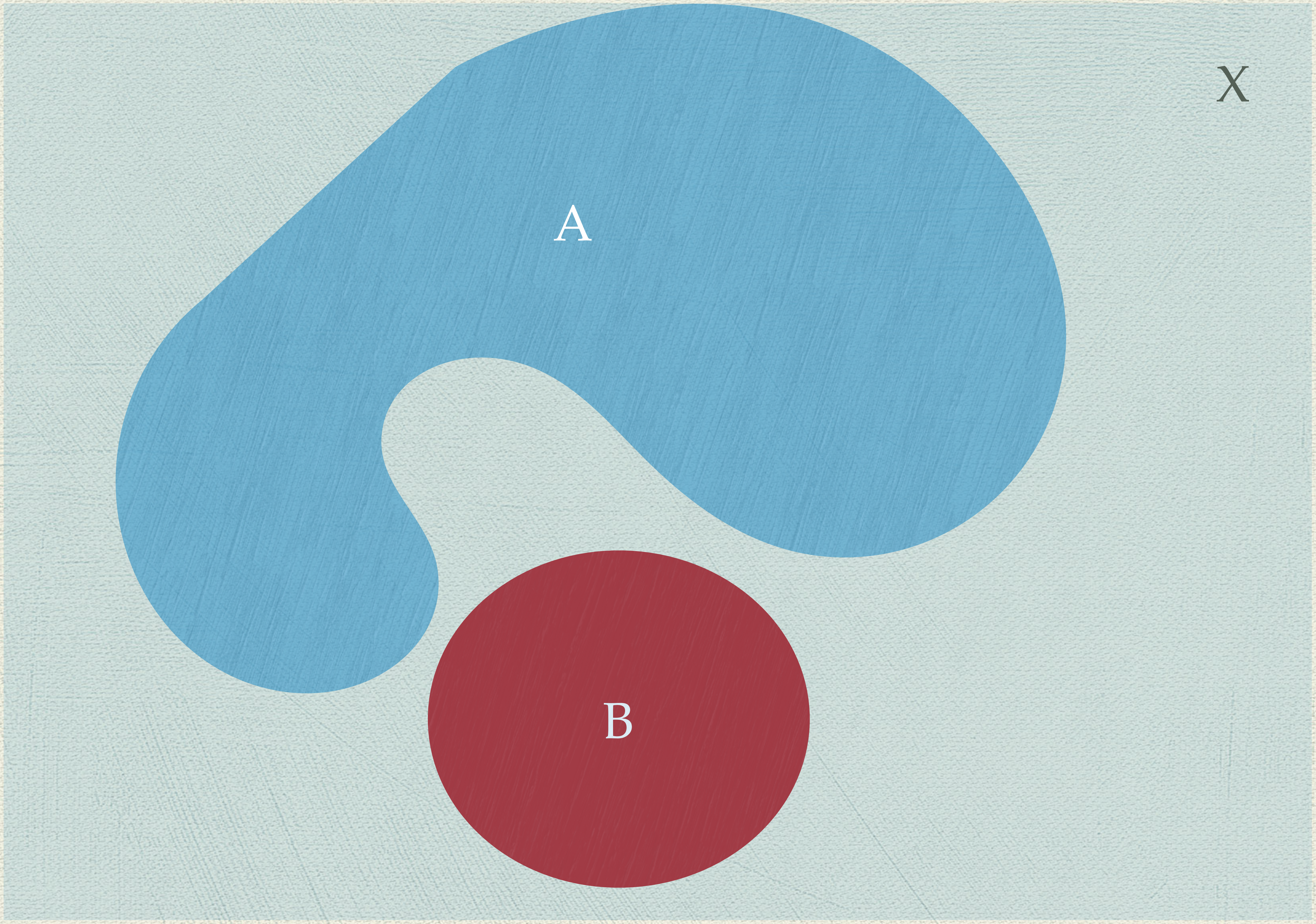
Experimental Verification of Entanglement

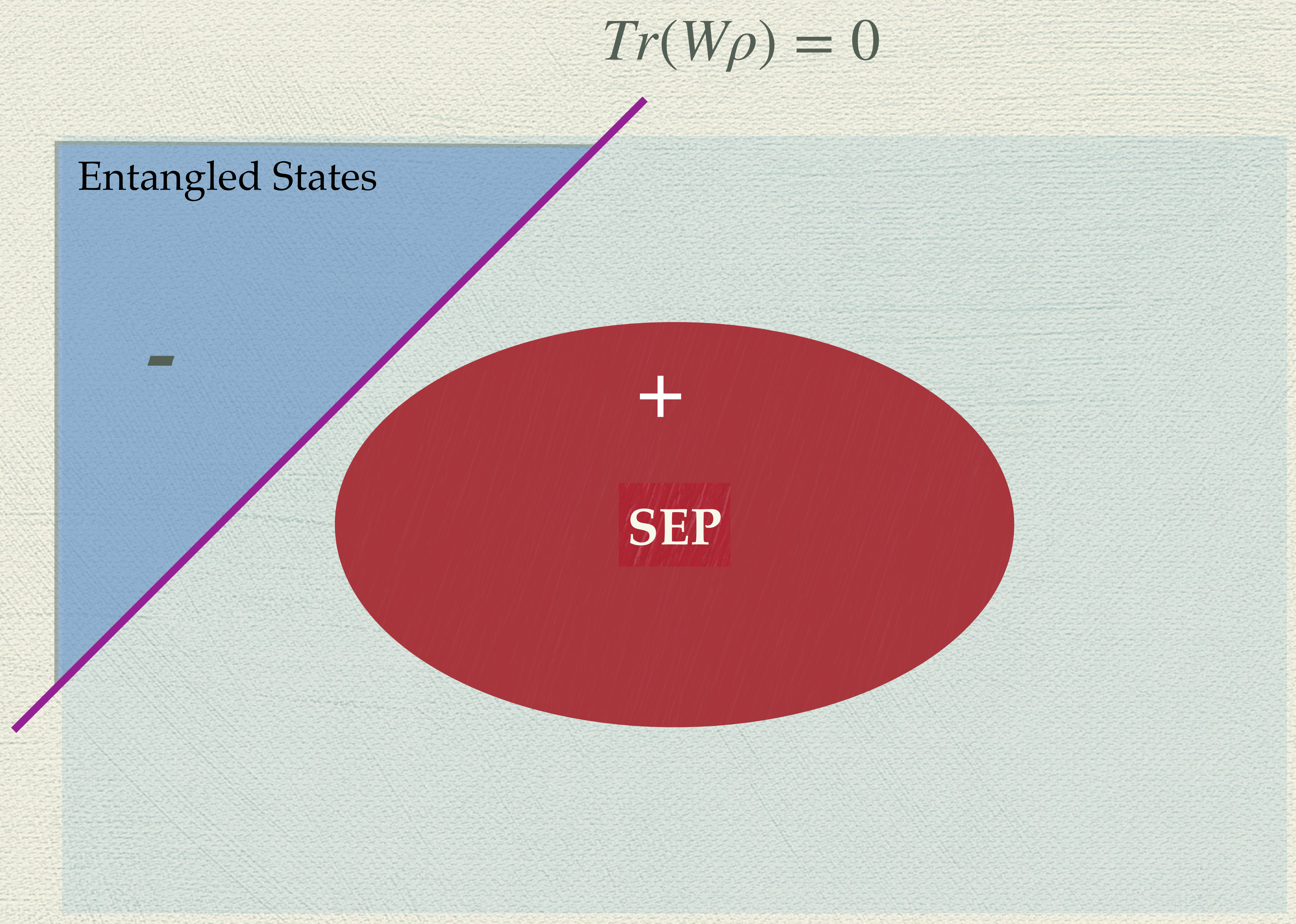
$Tr(W\rho) \longrightarrow$ Statement about entanglement

Hahn-Banach Theorem

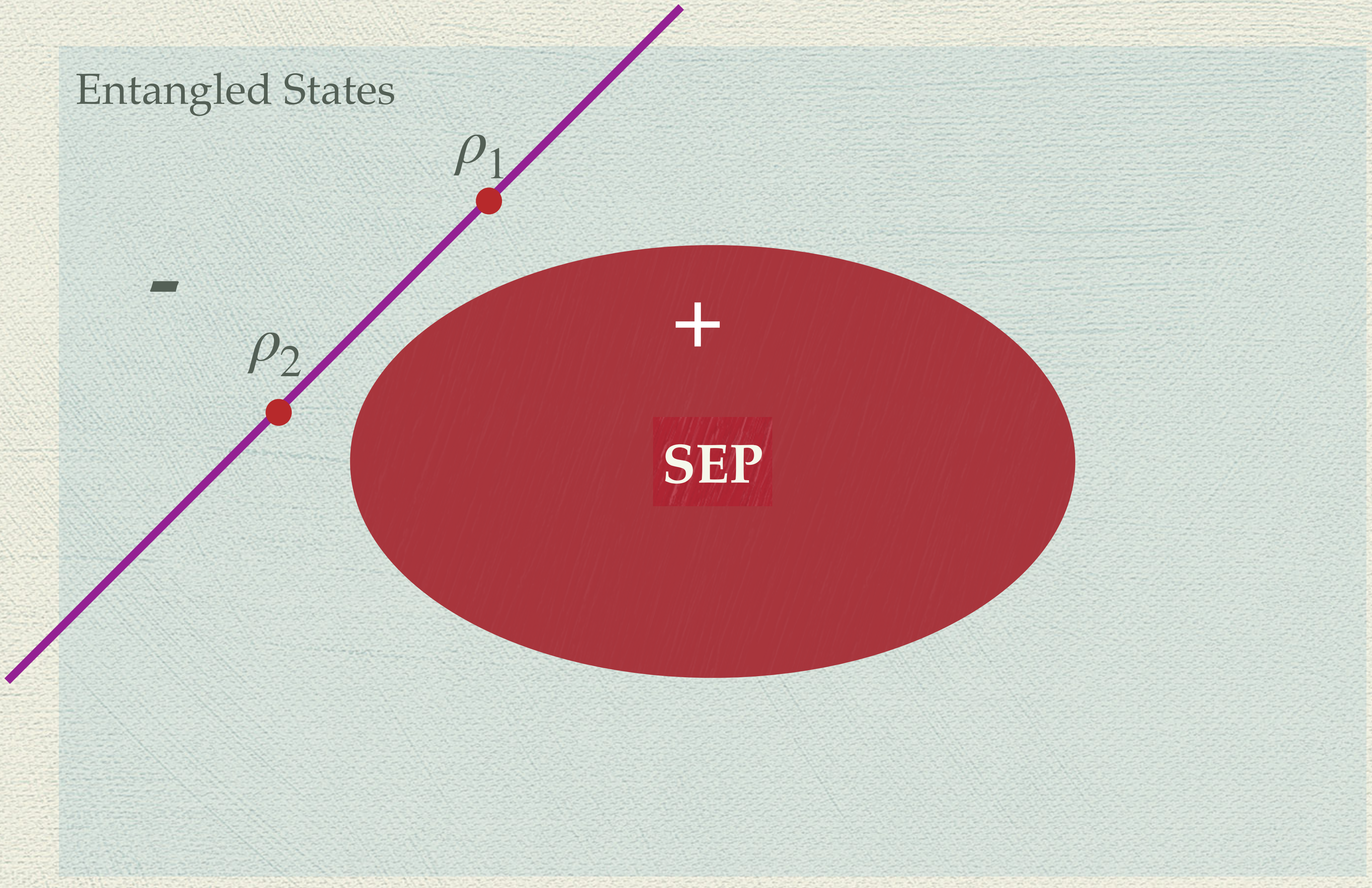


Hahn-Banach Theorem

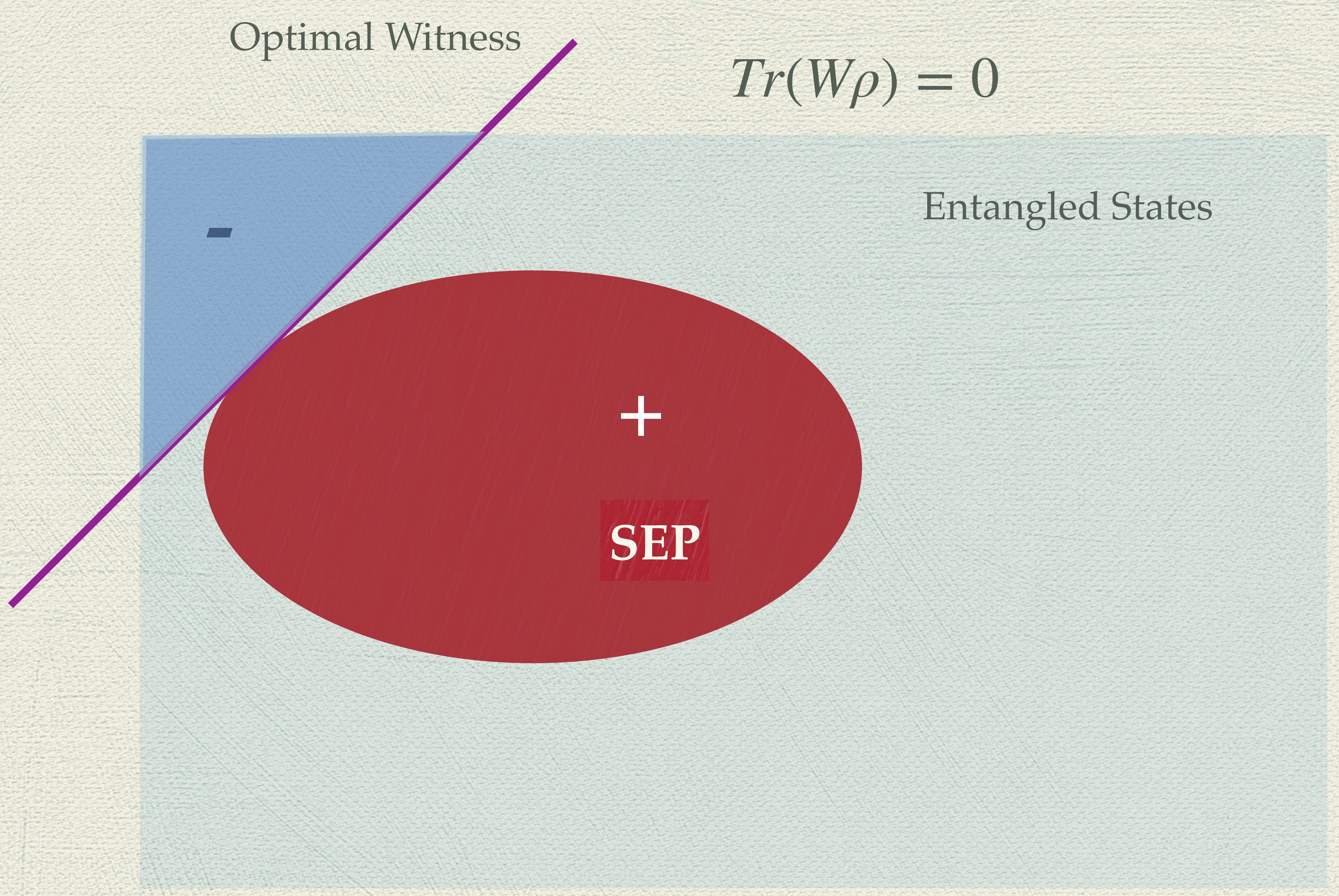



$Tr(W\rho) = 0$
Entangled States
-
+
SEP

$$Tr(W\rho) = 0$$

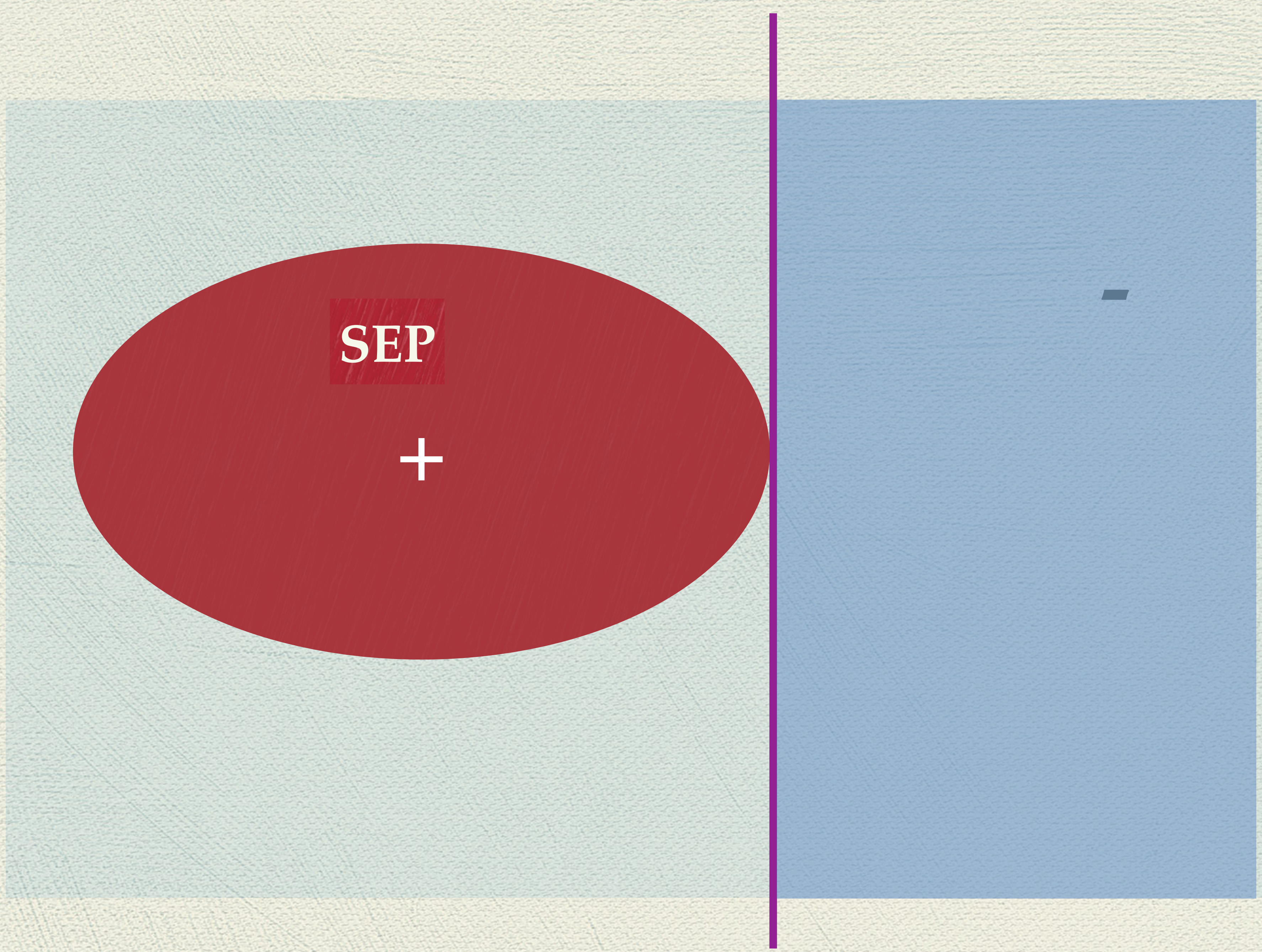


$Tr(W\rho_1) = 0$ $\quad$ $Tr(W\rho_2) = 0$ $\quad$ $Tr(W(\alpha\rho_1 + \beta\rho_2)) = 0$

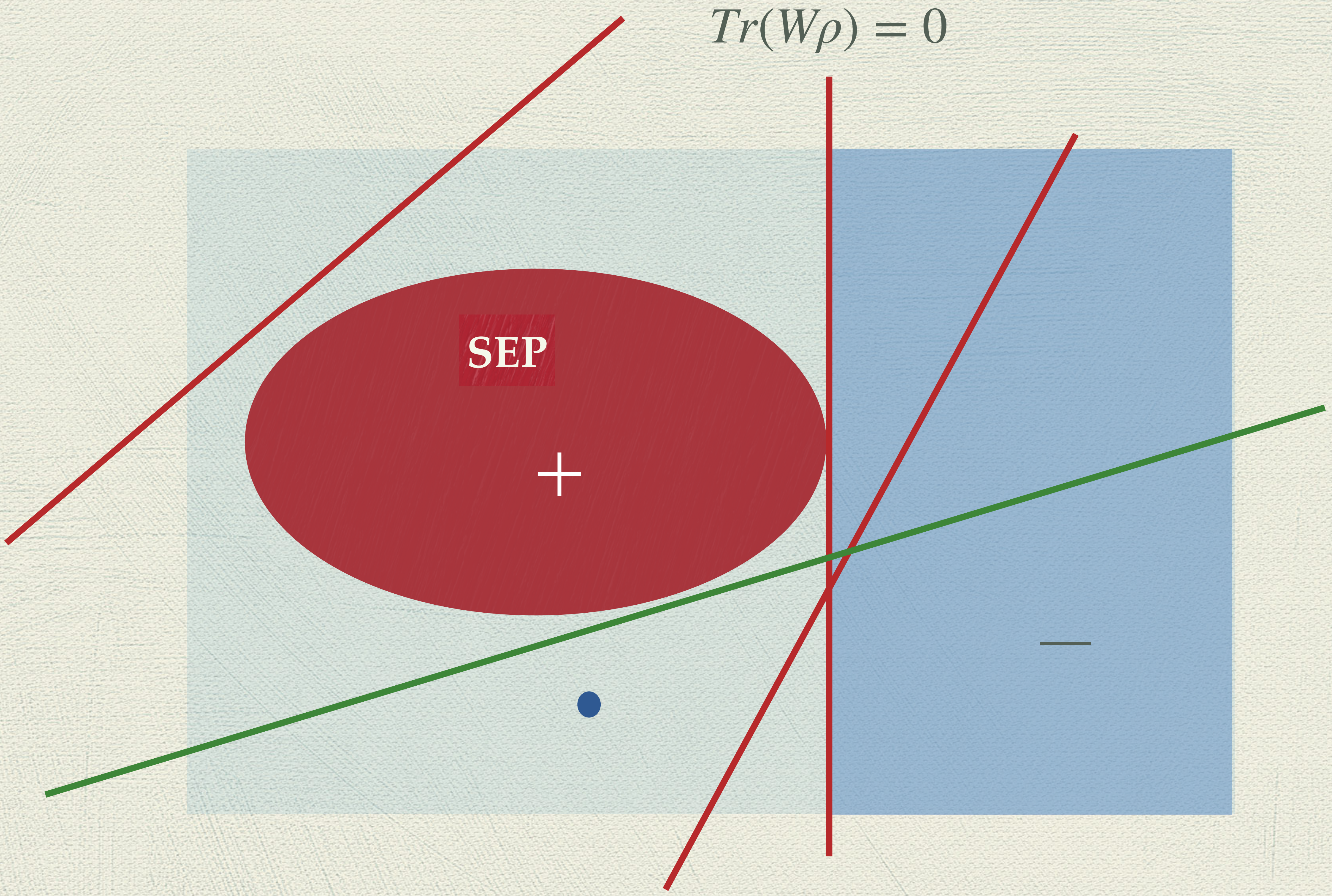


Optimal Witness
Tr(Wρ) = 0
Entangled States
-
+
SEP

$Tr(W\rho) = 0$



A good Witness



A good Witness

How to construct entanglement witnesses?

Let us learn from the PPT.

T=Transpose is a positive map

But it is not a Completely Positive Map (CPT)

$$(I \otimes T)|\phi^+\rangle\langle\phi^+| \ngeq 0$$

$$(I \otimes T)|\phi^+\rangle\langle\phi^+| \ngeq 0$$

$$W = (I \otimes T)|\phi^+\rangle\langle\phi^+|$$

$$= (I \otimes T)|i,i\rangle\langle j,j|$$

$$= |i\rangle\langle j| \otimes |j\rangle\langle i|$$

$$= |i,j\rangle\langle j,i| = P$$

$$Tr(W\rho^{sep}) > 0$$

But T is not the only positive map which is not CPT!

Let us take a map like Φ

$$W = (I \otimes \Phi)\,|\phi^+\rangle\langle\phi^+|$$

$$= \sum_{i,j} (I \otimes \Phi)\,|i,i\rangle\langle j,j|$$

$$= \sum_{i,j} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|)$$

$$W = \sum_{i,j} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|)$$

$$Tr\Big[W(\rho \otimes \sigma)\Big]$$

$$= \sum_{i,j} Tr|i\rangle\langle j|\rho \otimes Tr\ \Phi(|i\rangle\langle j|)\sigma$$

$$= \sum_{i,j} \rho_{j,i} Tr[\Phi(|i\rangle\langle j|)\sigma]$$

$$= Tr[\Phi(\rho^T)\sigma] \geq 0$$

Note: If A and B are positive, AB is not necessarily positive!

But

$$Tr(AB) \geq 0$$

$$Tr(AB) = \sum_{\alpha} \langle \alpha | B | \alpha \rangle \geq 0$$

To construct Witnesses,
find Positive Maps Φ which are not CTP

Then your witness will be

$$W = (I \otimes \Phi)|\phi^+\rangle\langle\phi^+|$$

$$= \sum_{i,j} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|)$$

Question: Is any witness of the form $W = (I \otimes \Phi)|\phi^+\rangle\langle\phi^+|$?

Yes:

W is a witness ⟷ $W = (I \otimes \Phi)|\phi^+\rangle\langle\phi^+|$

Proof:

$$W = \sum_{i,j} |i\rangle\langle j| \otimes \hat{W}_{ij}$$

$$W(|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|) \geq 0$$

$$\alpha_j^* \alpha_i \hat{W}_{ij}(|\beta\rangle\langle\beta|) \geq 0$$

Define: $\Phi(|i\rangle\langle j|) := \hat{W}_{ij}$

$$\Phi(|\alpha\rangle\langle\alpha|) := \alpha_i\alpha_j^*\hat{W}_{ij}$$

$$\langle\beta|\Phi(|\alpha\rangle\langle\alpha|)|\beta\rangle := \alpha_i\alpha_j^*\langle\beta|\hat{W}_{ij}|\beta\rangle \geq 0$$

Therefore Φ is a positive operator:

$$W = \sum_{i,j}|i\rangle\langle j| \otimes \hat{W}_{ij} = \sum_{i,j}|i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|)$$

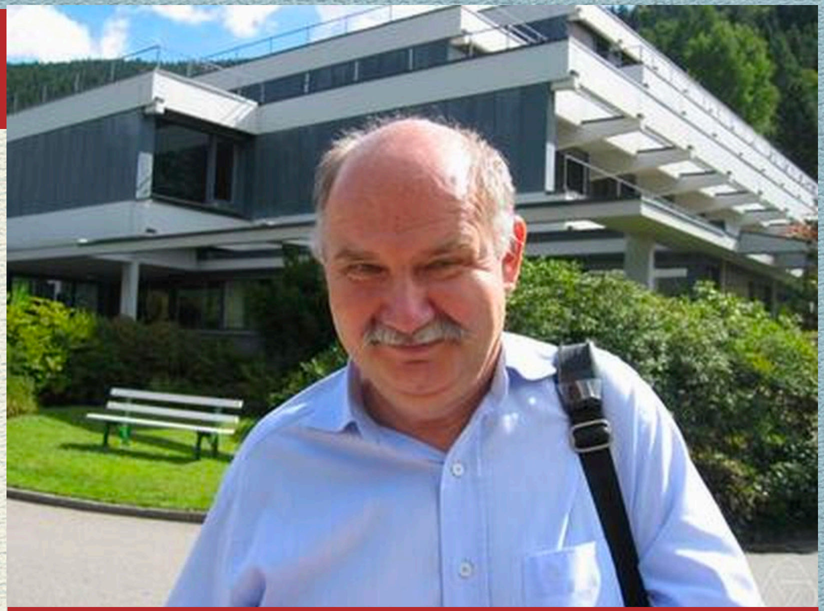
Classification of positive maps = Classification of Entanglement Witnesses

The problem:

Classification of CPT's is easy.

But classification of positive maps is difficult,

A basic theorem in 2*2 and 2*3 dimensions



S.L. Woronowics (1976)

Only if:

$$\Phi : L(H_2) \longrightarrow L(H_2)$$

$$\Phi : L(H_2) \longrightarrow L(H_3)$$

Every Positive maps is of the form: $\Phi = \mathscr{E}_1 + \mathscr{E}_2 \circ \mathscr{T}$

Where $\mathscr{T}$ is the transpose map.

Decomposable maps and their corresponding witnesses.

$$\mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2 \circ \mathscr{T} \longrightarrow W = P + (I \otimes T)Q$$

$$W = P + Q^{\Gamma}$$

Proof:

$$\mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2 \circ \mathscr{T}$$

$$W = \left[I \otimes \mathscr{E}_1 + I \otimes \mathscr{E}_2 \circ \mathscr{T} \right] (|\phi^+\rangle\langle\phi^+|)$$

$$\left[I \otimes \mathscr{E}_1 \right] (|\phi^+\rangle\langle\phi^+|) = P$$

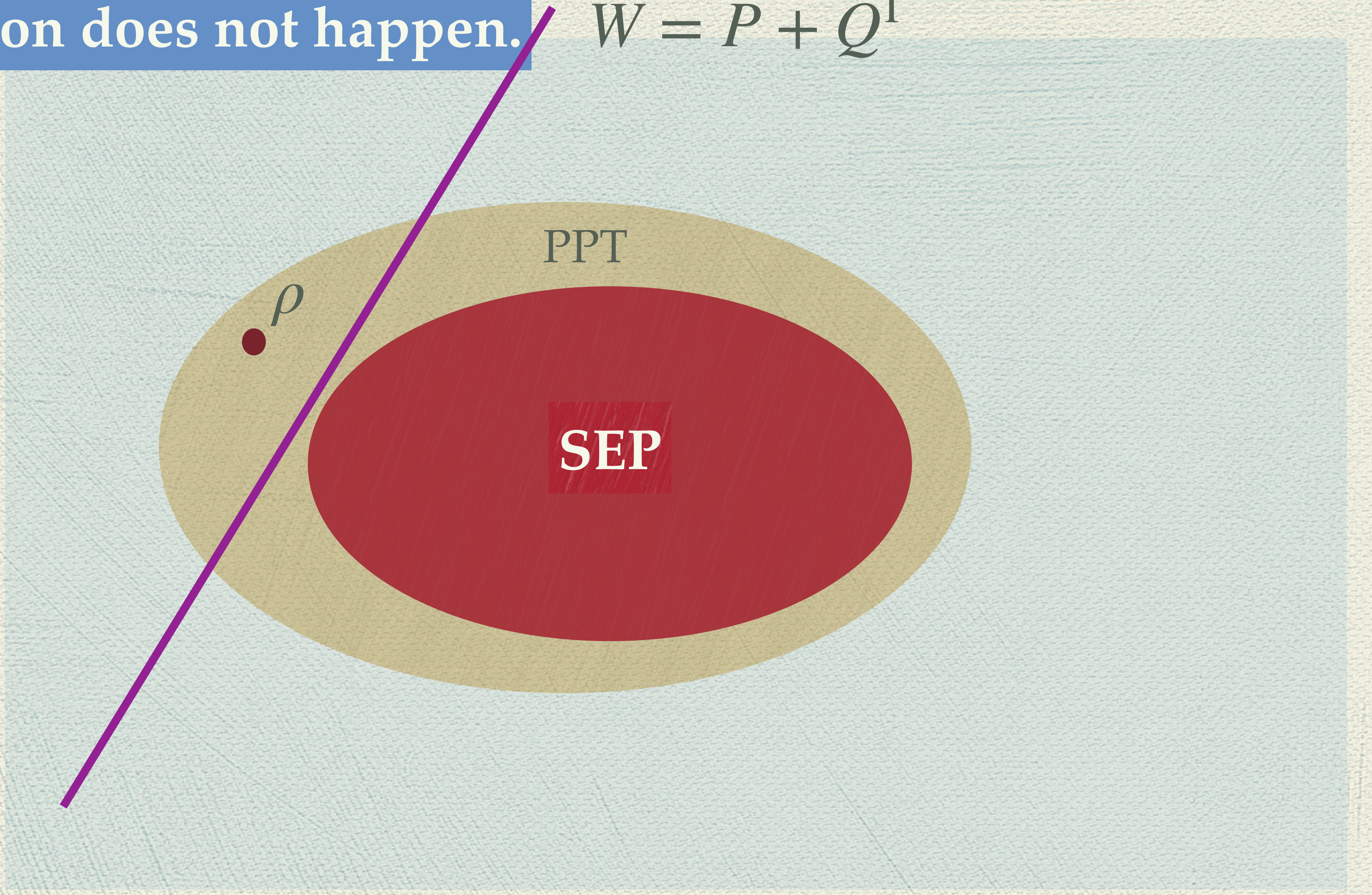
$$\left[I \otimes \mathscr{E}_2 \circ \mathscr{T} \right] (|\phi^+\rangle\langle\phi^+|) = (I \otimes T)Q$$

Why decomposable maps are important?

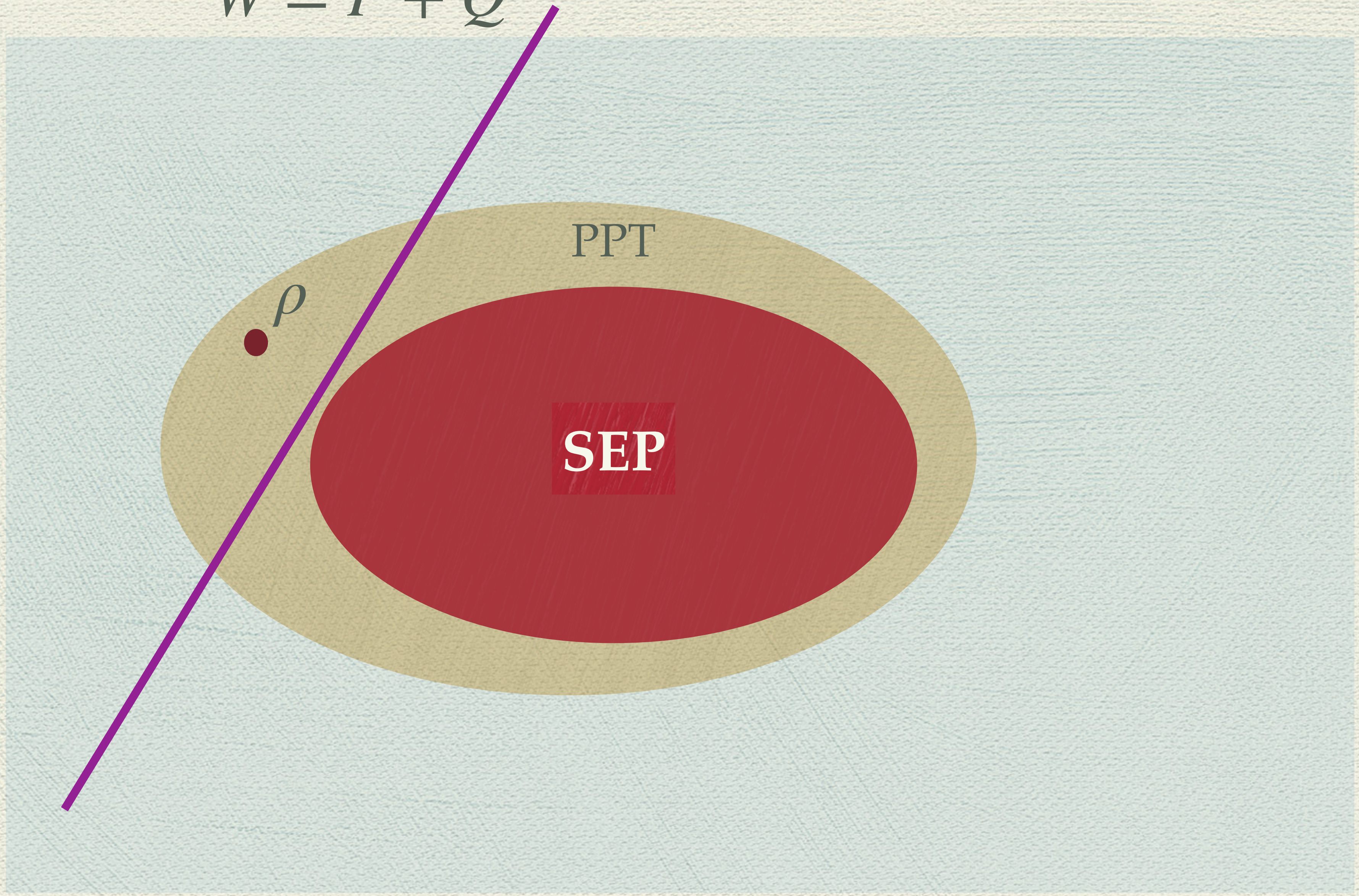
Because they detect PPT states.

This situation does not happen.

$W = P + Q^{\Gamma}$



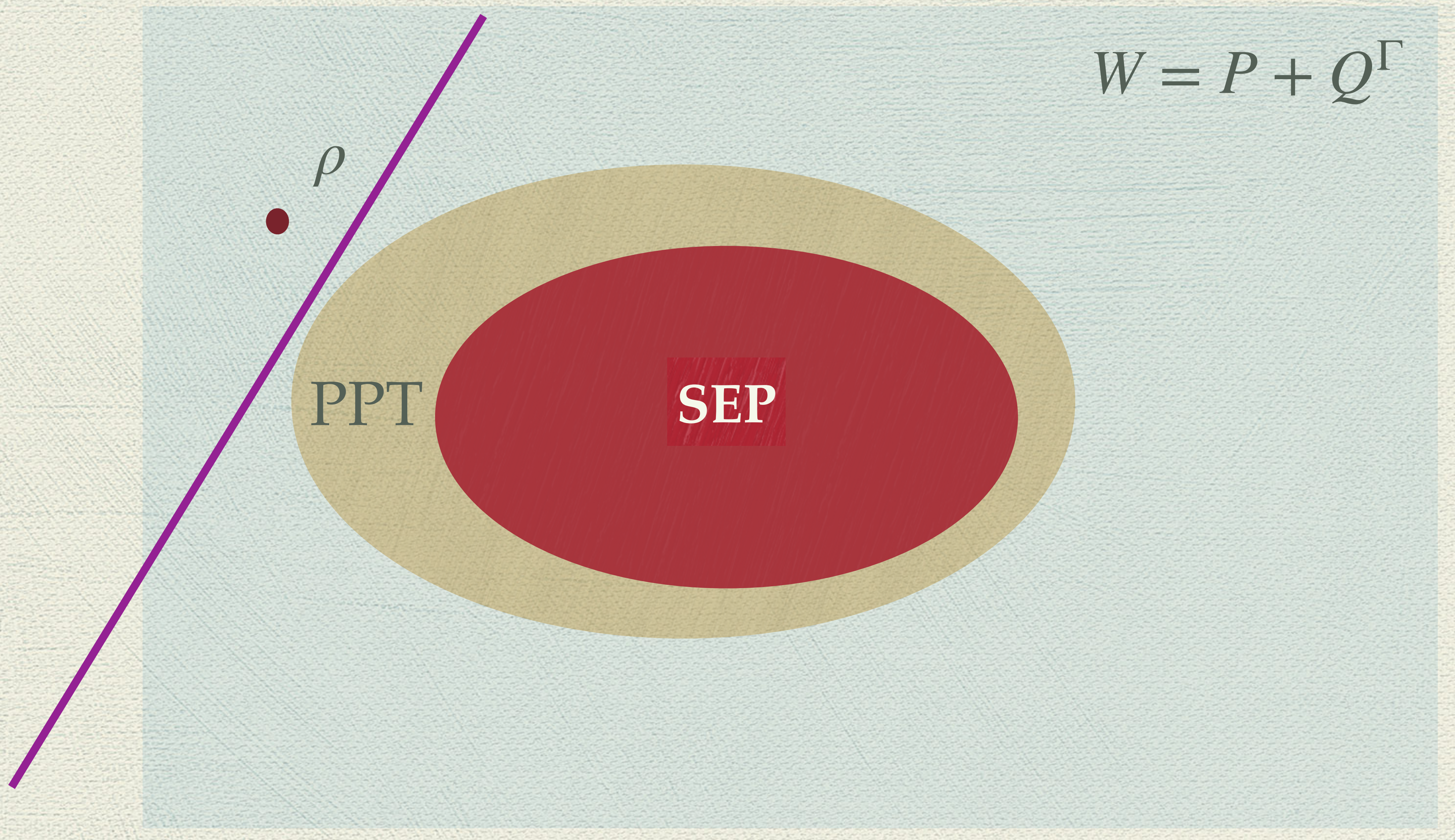
$W = P + Q^\Gamma$



We ask if this is possible $Tr(W\rho) < 0$?

But $Tr(W\rho) = Tr(P\rho) + Tr(Q^\Gamma \rho)$ $Tr(Q^\Gamma \rho) = Tr(Q\rho^\Gamma) > 0$

This situation happens.



So if $Tr(W\rho) < 0$ $\longrightarrow$ ρ is not PPT!

This is the proof that :

In 2×2 *and* 2×3 *dimensions*

PPT=SEP

Horodecki Family

What about higher dimensions?

A journey to the 19th Century and basic mathematics.

Every Positive quadratic form is the sum of squares

$$a^2 + b^2 + 2ab = (a+b)^2$$

$$2a^2 + 5b^2 - 2ab = (a+b)^2 + (a-2b)^2$$

$$Q_2(\mathbf{x}) = \sum_{\alpha} |Q_1^{\alpha}(\mathbf{x})|^2$$

$$Q_2(\mathbf{x}) = \sum_{\alpha} |Q_1^{\alpha}(\mathbf{x})|^2$$

$$Q_4(\mathbf{x}) = \sum_{\alpha} |Q_2^{\alpha}(\mathbf{x})|^2 \quad ?$$

$$Q_6(\mathbf{x}) = \sum_{\alpha} |Q_3^{\alpha}(\mathbf{x})|^2 \quad ?$$

David Hilbert: (1888)

David Hilbert (1862-1943)

Every quartic polynomial of 3 variables

$$Q_4(x, y, z) = \sum_{\alpha=1}^{3} [Q_2^{(\alpha)}(x, y, z)]^2$$

Hilbert (1888): For all other combinations of (n>3 , d) the answer is negative.

Number of variables

Degree of polynomials

Quadratic forms and operators

$$Q(\mathbf{x}) = \sum_{i,j} \bar{x}_i Q_{ij} x_j$$

$$\hat{Q} : H_n \longrightarrow H_n$$

$$|y_i\rangle = \sum_j \hat{Q}_{ij} |x_j\rangle$$

If the quadratic form Q(x) is positive, i.e. If $\langle \mathbf{x} | \hat{Q} | \mathbf{x} \rangle \geq 0$

We can rewrite Q(x) in the following form:

$$Q(\mathbf{x}) = \sum_{\alpha} |(\mathbf{b}_{\alpha} \cdot \mathbf{x})|^2$$

$$\langle \mathbf{x} | \hat{Q} | \mathbf{x} \rangle = \sum_{\alpha} \langle \mathbf{x} | \mathbf{b}_{\alpha} \rangle \langle \mathbf{b}_{\alpha} | \mathbf{x} \rangle$$

$$\longrightarrow \quad \hat{Q} = \sum_{\alpha} | \mathbf{b}_{\alpha} \rangle \langle \mathbf{b}_{\alpha} |$$

A positive linear operator corresponds to this quadratic form.

Bi-quadratic Polynomials

$$\Phi(\mathbf{x}, \mathbf{y}) = \sum_{i,j;k,l} \bar{y}_i y_j \Phi_{ij;kl} x_k \bar{x}_l$$

$$x^2 y^2 = (xy)^2$$

$$a^2x^2 + 9b^2y^2 + a^2y^2 + 4b^2x^2 - 2abxy = (ax - 3by)^2 + (ay + 2bx)^2$$

Bi-linear

Is this general?

Bi-quadratic forms and Super-operators (maps).

$$\Phi(\mathbf{x}, \mathbf{y}) = \sum_{i,j;k,l} \bar{y}_i y_j \Phi_{ij;kl} x_k \bar{x}_l$$

$$\hat{\Phi} : L(H_n) \longrightarrow L(H_n)$$

$$\hat{\Phi} : |x_k\rangle\langle x_l| \longrightarrow \sum_{i,j} \Phi_{ij;kl} |x_k\rangle\langle x_l|$$

$$\hat{\Phi} : \rho_{kl} \longrightarrow \sum_{i,j} \Phi_{ij;kl} \rho_{kl}$$

If $\Phi(\mathbf{x}, \mathbf{y})$ is a positive bi-quadratic form

→ $\hat{\Phi}$ is a positive map.

Proof: Let ρ be a positive matrix $\rho_{kl} = \sum_{\alpha} x_k^{\alpha} \bar{x}_l^{\alpha}$

$$\langle \mathbf{y} | \hat{\Phi}(\rho) | \mathbf{y} \rangle = \sum_{i,j} \bar{y}_i \hat{\Phi}(\rho)_{ij} y_j$$

$$= \sum_{\alpha} \sum_{i,j} \bar{y}_i y_j \Phi_{ij,kl} x_k^{\alpha} \bar{x}_l^{\alpha}$$

Question: What corresponds to bi-quadratic forms Which is a sum of squares of bi-linear forms?

$$\Phi(\mathbf{x},\mathbf{y}) = \sum_{\alpha} |L^{\alpha}(\mathbf{x},\mathbf{y})|^2$$

$$L(\mathbf{x},\mathbf{y}) = \sum_{i,l} \bar{y}_i L_{il} x_l \qquad ?$$

$$\Phi(\mathbf{x},\mathbf{y}) = \sum_{\alpha} \sum_{i,j,k,l} \bar{y}_i L^{\alpha}_{ik} x_k \quad y_j \overline{L^{\alpha}_{jl}} \bar{x}_l$$

$$\Phi_{ijkl} = \sum_{\alpha} L^{\alpha}_{ik} \overline{L^{\alpha}_{jl}}$$

$$\hat{\Phi} : \rho_{kl} \longrightarrow \sum_{i,j} \Phi_{ij;kl}\rho_{kl}$$

$$\hat{\Phi} : \rho_{kl} \longrightarrow \sum_{i,j} \sum_{\alpha} L^{\alpha}_{ik}\overline{L^{\alpha}_{jl}}\rho_{kl}$$

$$\hat{\Phi} : \rho \longrightarrow \sum_{\alpha} L^{\alpha}\rho L^{\alpha\dagger}$$

Therefore if every positive bi-quadratic form is the sum of squares of bi-linear forms

Then every positive map is a CPT.

Question: What corresponds to bi-quadratic forms Which is a sum of squares of bi-linear forms?

Choi (1975): **The answer is NO.**

Positive Semidefinite Biquadratic Forms

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Submitted by Chandler Davis

Counter-Example:

$$F = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$$

$$-2(x_1 x_2 y_1 y_2 + x_2 x_3 y_2 y_3 + x_3 x_1 y_3 y_1)$$

$$+2(x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2)$$

So every positive map Φ is not a CPT $\mathscr{E}$.

What about

$$\Phi = \mathscr{E}_1 + \mathscr{E}_2 \circ \mathscr{T}$$

Again the answer is NO!

If $\quad \hat{\Phi} = \mathscr{E}_1 + \mathscr{E}_2 \circ \mathscr{T}$

$$\hat{\Phi}(\rho) = \sum_\alpha F_\alpha \rho F_\alpha^\dagger + \sum_\beta G_\beta \rho^T G_\beta^\dagger$$

$$\Phi_{ij,kl}\rho_{kl} = \sum_\alpha (F_\alpha)_{ik}\rho_{kl}(F_\alpha^\dagger)_{lj} + \sum_\beta (G_\beta)_{ik}\rho_{lk}(G_\beta^\dagger)_{lj}$$

$$\Phi_{ij,kl} = \sum_\alpha (F_\alpha)_{ik}(F_\alpha^\dagger)_{lj} + \sum_\beta (G_\beta)_{il}(G_\beta^\dagger)_{kj}$$

$$\Phi_{ij,kl} = \sum_{\alpha} (F_\alpha)_{ik}(F_\alpha^\dagger)_{lj} + \sum_{\beta} (G_\beta)_{il}(G_\beta^\dagger)_{kj}$$

$$\Phi_{ij,kl} = \sum_{\alpha} (F_\alpha)_{ik}\overline{(F_\alpha)_{jl}} + \sum_{\beta} (G_\beta)_{il}\overline{(G_\beta)_{jk}}$$

$$\begin{aligned}
\sum_{ijkl} y_i\bar{y}_j\Phi_{ij,kl}x_k\bar{x}_l &= \sum_{\alpha} y_i\bar{y}_j(F_\alpha)_{ik}\overline{(F_\alpha)_{jl}}\; x_k\bar{x}_l \\
&+ \sum_{\beta} y_i\bar{y}_j(G_\beta)_{il}\overline{(G_\beta)_{jk}}x_k\bar{x}_l \\
&= \sum_{\alpha} |(\mathbf{y}^T F^\alpha \mathbf{x})|^2 \\
&+ \sum_{\beta} |(\mathbf{y}^T G^\beta \overline{\mathbf{x}})|^2
\end{aligned}$$

Therefore if

$$\hat{\Phi} = \mathscr{E}_1 + \mathscr{E}_2 \circ \mathscr{T}$$

We conclude that

$$\Phi(\mathbf{x}, \mathbf{y}) \;\; = \sum_{\alpha} |(\mathbf{y}^T F^{\alpha} \mathbf{x})|^2 + \sum_{\beta} |(\mathbf{y}^T G^{\beta} \overline{\mathbf{x}})|^2$$

which we know is not always true.

Summary:

For qubits-qubit and qubit-qutrit, any Witness is of the form

$$W = P + Q^{\Gamma}$$

In higher dimensions, there are other forms of witnesses.

End of Part I